In this chapter you will:

- prove two triangles are congruent
- learn about line symmetry and rotational symmetry
- learn about some special types of 2D shapes
- recognise similar shapes and use scale factors to find missing sides in similar triangles
- formally prove that triangles are similar.

The photo shows the Pyramide du Louvre in Paris. There are actually five pyramids, the large one, three smaller ones and an inverted pyramid which provides the entrance to the Louvre museum. The larger pyramid is made up of 603 diamond-shaped panes of glass with 70 triangular-shaped panes along the base of the pyramid.

Objectives

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- formally prove that triangles are similar.

Before you start

You need to:

- know the angle properties of triangles and quadrilaterals
- know what a vertex and a diagonal of a shape are.
8.1 Congruent triangles

Objective

- You will understand how to prove that two triangles are congruent.

Why do this?

Designers, engineers and map makers often use scale drawings and plans. Using congruent and similar triangles enables them to find measurements for inaccessible lengths and angles.

Get Ready

1. If two triangles have the same angles, are the triangles the same?
2. Given the lengths of all three sides, is it possible to draw two different triangles?
3. Given lengths of two sides and the size of the included angle, is it possible to draw two different triangles?

Key Points

- Two triangles are congruent if they have exactly the same shape and size.
- For two triangles to be congruent one of the following conditions of congruence must be true.
  - The three sides of each triangle are equal (SSS).
  - Two sides and the included angle are equal (SAS).
  - Two angles and a corresponding side are equal (AAS).
  - Each triangle contains a right angle, and the hypotenuses and another side are equal (RHS).

Example 1

ABCD is a quadrilateral.
AD = BC.
AD is parallel to BC.
Prove that triangle ADC is congruent to triangle ABC.

Each statement for a congruence proof must be justified.

ResultsPlus

Watch Out!

The only properties that can be used to prove congruence are those given in the question.

AD = BC (given)
Angle DAC = angle ACB (alternate angles)
AC is common to both triangles.
Hence triangle ADC is congruent to triangle ABC
(two sides and the included angle).
Chapter 8 Congruence, symmetry and similarity

Exercise 8A

1. Prove that triangles PQS and QRS are congruent.

2. Prove that triangles XYZ and XVW are congruent. Hence prove that X is the midpoint of YW.

3. PQR is an isosceles triangle. S and T are points on QR. \( PQ = PR, QS = TR. \) Prove that triangle PST is isosceles.

4. LMN is an isosceles triangle with LM = LN. Use congruent triangles to prove that the line from L which cuts the base MN of the triangle at right angles also bisects the base.

5. ABC is a triangle. D is the midpoint of AB. The line through D drawn parallel to the side BC meets the side AC at E. A line through D drawn parallel to the side AC meets the side BC at F. Prove that triangles ADE and DBF are congruent.

8.2 Symmetry in 2D shapes

Objectives

- You can recognise line symmetry in 2D shapes.
- You can draw lines of symmetry on 2D shapes.
- You can recognise rotational symmetry in 2D shapes.
- You can find the order of rotational symmetry of a 2D shape.
- You can draw shapes with given line symmetry and/or rotational symmetry.

Why do this?

There are examples of 2D symmetry in the man-made and natural world, such as wheels, flowers and butterflies.

Get Ready

1. Trace this star.
   Fold your tracing along the dotted line. What do you notice? Place your tracing over the star and turn the tracing paper clockwise. Keep turning the tracing paper until you get back to the starting position. What do you notice?
A shape has **line symmetry** if it can be folded so that one part of the shape fits exactly on top of the other part.

Every point of the shape on one side of the line of symmetry has a corresponding point on the mirror image the other side of the line. Notice that the point A and its corresponding point B are the same distance from the line of symmetry.

If a mirror were placed on the line of symmetry, the shape would look the same. This is why line symmetry is sometimes called **reflection symmetry** and the line of symmetry is sometimes called the **mirror line**.

A shape has **rotation symmetry** if a tracing of the shape fits exactly on top of the shape in more than one position when it is rotated.

A tracing of a shape with rotation symmetry will fit exactly on top of the shape when turned through less than a complete turn.

The number of times that the tracing fits exactly on top of the shape is called the **order of rotational symmetry**.

Some two-dimensional shapes do not have any symmetry.

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**Example 2**

Draw in all the lines of symmetry on this flag.

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**Example 3**

Find the order of rotational symmetry of this shape.

The shape has rotational symmetry of order 3.
**Example 4**
Copy and complete the drawing of the shape so that it has line symmetry.

![Diagram of a shape with line of symmetry indicated.]

*Each corner has a mirror image the same distance away from the line of symmetry but on the other side of it.*

*Mark the mirror images of the corners on the other side of the line of symmetry.*

*Then complete the shape.*

**Example 5**
Shade four more squares in the pattern so that the dotted line is a line of symmetry.

![Diagram of a shaded pattern with line of symmetry indicated.]

*Compare the two sides of the pattern.*

*Shade four squares so that each shaded square has an image on the other side of the line of symmetry.*

**Example 6**
The diagram shows part of a shape.
Complete the shape so that it has no lines of symmetry and rotational symmetry of order 2.

![Diagram of a shape with rotational symmetry indicated.]

**Examiner’s Tip**
Rotational symmetry of order 2 means that the completed shape must look the same when it is rotated through a half turn.
1. For each shape, write down if it has line symmetry or not. If it has symmetry, copy the diagram and draw in all the lines.

   a
   b
   c
   d
   e
   f

2. Using tracing paper if necessary, state which of the following shapes have rotational symmetry and which do not have rotational symmetry.

   For the shapes that have rotational symmetry, write down the order of the rotational symmetry.

   a
   b
   c
   d
   e
   f

3. a. Copy and complete this shape so that it has line symmetry.
   b. Write down the name of the complete shape.

4. Each diagram shows an incomplete pattern.
   For part a, copy the diagram and shade six more squares so that both dotted lines are lines of symmetry of the complete pattern. For part b, shade three more squares so that the complete pattern has rotational symmetry of order 4.

   a
   b
Chapter 8 Congruence, symmetry and similarity

8.3 Symmetry of special shapes

Objectives

- You know the symmetries of special triangles.
- You can recognise and name special quadrilaterals.
- You know the properties of special quadrilaterals.
- You know the symmetries of special quadrilaterals.
- You know the symmetries of regular polygons.

Why do this?

Many architectural designs are symmetrical in some way. The Taj Mahal, the Pyramids and the Greek Parthenon have impressive and beautiful uses of symmetry.

Get Ready

1. a What is i an isosceles triangle ii an equilateral triangle?
   b Is an equilateral triangle an isosceles triangle?

2. What is a quadrilateral?

Key Points

Triangles

A triangle is a polygon with three sides. Here is an isosceles triangle. It has two sides the same length. An isosceles triangle has one line of symmetry.

Here is an equilateral triangle. All its sides are the same length. An equilateral triangle has three lines of symmetry and rotational symmetry of order 3.

Quadrilaterals

A quadrilateral is a polygon with four sides. Some quadrilaterals have special names. Here are some of the properties of special quadrilaterals.

Square

All sides equal in length.
All angles are 90°.
4 lines of symmetry and rotational symmetry of order 4.

Rectangle

Opposite sides equal in length.
All angles are 90°.

Rhombus

All sides equal in length.
Opposite sides parallel.
Opposite angles equal.
2 lines of symmetry and rotational symmetry of order 4.
8.4 Recognising similar shapes

Objectives

- You can recognise similar shapes.
- You can find missing sides using facts you know about similar shapes.

Why do this?

Similar shapes allow us to calculate missing dimensions from plans which may be difficult to measure on the real objects.

Get Ready

1. Which of these triangles are congruent?

- **A**
  - Base: 10 cm
  - Side: 5.8 cm
  - Angle: 30°

- **B**
  - Base: 15 cm
  - Side: 8.7 cm
  - Angle: 30°

- **C**
  - Base: 10 cm
  - Side: 6 cm
  - Angle: 60°

Exercise 8C

1. a On squared paper, draw a right-angled triangle that has one line of symmetry. Draw the line of symmetry on your triangle.
   b Write down what is special about this right-angled triangle.

2. Janine says, ‘I am thinking of a quadrilateral. It has opposite sides that are parallel.’
   a Is there enough information to know what the quadrilateral is? Give reasons for your answer.
   Janine now says, ‘It has rotational symmetry of order 2.’
   b Is there now enough information to know what the quadrilateral is? Give reasons for your answer.
   Janine now says, ‘It has two lines of symmetry.’
   c Is there now enough information to know what the quadrilateral is? Give reasons for your answer.
   Janine now says, ‘It has sides that are not all the same length.’
   d What quadrilateral is Janine thinking of?

3. Draw a non-regular polygon which has a line of symmetry.

4. Draw a non-regular polygon which has rotational symmetry. State the order of rotational symmetry of your polygon.
**Chapter 8 Congruence, symmetry and similarity**

**Key Points**

- Shapes are **similar** if one shape is an enlargement of the other.
- The corresponding angles are equal.
- The corresponding sides are all in the same ratio.

**Example 7**

Show that the parallelogram ABCD is not similar to parallelogram EFGH.

\[
\begin{align*}
\frac{AB}{EF} & = \frac{3}{4} = 0.75 \\
\frac{BC}{FG} & = \frac{6}{9} = 0.66667
\end{align*}
\]

The lengths of the corresponding sides are not in the same proportion so the parallelograms are not similar.

**Exercise 8D**

1. State which of the pairs of shapes are similar.
   - a) 1.5 cm, 3 cm, 1 cm, 2 cm
   - b) 2 cm, 3 cm, 3 cm, 6 cm

2. Show that pentagon ABCDE is similar to pentagon FGHJ.
Example 8

These two rectangles are similar. Find the length \( L \) of the larger rectangle.

The widths of these rectangles are in the ratio 2 : 3.

The lengths must be in the same ratio.

\[
\frac{\text{small width}}{\text{large width}} = \frac{2}{3} = \frac{4}{L}
\]

\[
2L = 12
\]

\[
L = 6 \text{ cm}
\]

Exercise 8E

1. A large packet of breakfast cereal has height 35 cm and width 21 cm. A small packet of cereal is similar to the large packet but has a height of 25 cm. Find the width of the small packet.

2. The diagram shows a design for a metal part. The sizes of the plan are marked on diagram A. Diagram B is marked with the actual sizes. Calculate the value of:
   a. \( x \)
   b. \( y \)

3. These cylinders are similar. The height of the smaller cylinder is 5 cm. Find the height of the larger cylinder.
**8.5 Similar triangles**

**Objectives**
- You can use scale factors to find missing sides in similar triangles.
- You can formally prove that triangles are similar.

**Why do this?**
Using the fact that triangles are similar can help us to measure lengths and distances which we cannot measure practically.

**Get Ready**

1. Copy these diagrams and mark the pairs of corresponding angles.

![Diagrams](image)

**Key Points**
- Two triangles are similar if any of the following is true.
  - The corresponding angles are equal.
  - The corresponding sides are in the same ratio.
  - They have one angle equal and the adjacent sides are in the same ratio.

\[ \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} \]

**Example 9**
Show that triangle ABC is not similar to triangle DEF.

![Example Diagram](image)

\[ \frac{AB}{DE} = \frac{4}{5} = 0.8 \]
\[ \frac{AC}{DF} = \frac{3}{12} = 0.25 \]

The lengths of the corresponding sides are not in the same proportion so the triangles are not similar.

**Examiner’s Tip**
Make sure you look carefully at the parallel lines in a diagram, as they can give you a lot of information about the angles.
Example 10

ABC is a triangle.

DE is parallel to BC.

a Show that triangle ABC is similar to triangle ADE.
b Find the length of BD.

a \( \angle ADE = \angle ABC \) (corresponding angles).
\( \angle AED = \angle ACB \) (corresponding angles).
\( \angle DAE = \angle BAC \) (common to both triangles).
All angles are equal so triangle ABC is similar to triangle ADE.

b \[
\frac{BC}{DE} = \frac{8}{3} = \frac{AB}{2.5}
\]
\[
3 \times AB = 8 \times 2.5
\]
\[
3AB = 20
\]
\[
AB = 6.67 \text{ cm}
\]
\[
BD = 6.67 - 2.5 = 4.17 \text{ cm}
\]

Give reasons from what you know about parallel lines.

The corresponding sides are in the same ratio.

BD is only part of the side of the triangle.

Exercise 8F

1 For each pair of similar triangles:
i name the three pairs of corresponding sides
ii state which pairs of angles are equal.

2 Triangle ABC is similar to triangle DEF.
\( \angle ABC = \angle DEF \)
Calculate the length of:
a EF
b FD.
3. The diagram shows triangle ABC which has a line DE drawn across it.  
\[ \angle ACE = \angle DEB \]
- **a** Prove that triangle ABC is similar to triangle DBE.
- **b** Calculate the length of AB.
- **c** Calculate the length of AD.
- **d** Calculate the length of EC.

4. The diagram shows triangle ABC which has a line DE drawn across it.  
\[ \angle CAD = \angle BDE \]
- **a** Prove that triangle ACB is similar to triangle DEB.
- **b** Calculate the length of DE.
- **c** Calculate the length of BC.

5. In the diagram AB is parallel to CD.
   - **a** Prove that triangle ABM is similar to triangle CDM.
   - **b** AC has length 20 cm.
     - Calculate the lengths of:  
       - i AM  
       - ii MC.

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**Chapter review**

- Two triangles are **congruent** if they have exactly the same shape and size.
- For two triangles to be congruent one of the following **conditions of congruence** must be true.
  - The three sides of each triangle are equal (SSS).
  - Two sides and the **included angle** are equal (SAS).
  - Two angles and a corresponding side are equal (AAS).
  - Each triangle contains a right angle, and the hypotenuses and another side are equal (RHS).
- A shape has **line symmetry** if it can be folded so that one part of the shape fits exactly on top of the other part.
  - Every point of the shape on one side of the **line of symmetry** has a corresponding point on the **mirror image** the other side of the line. Notice that corresponding points are the same distance from the line of symmetry.
  - If a mirror were placed on the line of symmetry of a shape, the shape would look the same in the mirror. This is why line symmetry is sometimes called **reflection symmetry** and the line of symmetry is sometimes called the **mirror line**.
- A shape has **rotation symmetry** if a tracing of the shape fits exactly on top of the shape in more than one position when it is rotated.
  - A tracing of a shape with rotation symmetry will fit exactly on top of the shape when turned through less than a complete turn.
  - The number of times that the tracing fits exactly on top of the shape is called the **order of rotational symmetry**.
Some shapes do not have any symmetry.

Shapes are **similar** if one shape is an enlargement of the other.
- The corresponding angles are equal.
- The corresponding sides are all in the same ratio.

Two triangles are similar if any of the following is true.
- The corresponding angles are equal.
- The corresponding sides are in the same ratio.
- They have one angle equal and the adjacent sides are in the same ratio.

### Review exercise

1. **a** On the diagram below, shade **one** square so that the shape has exactly **one** line of symmetry.

![Diagram](image1.png)

**b** On the diagram below, shade **one** square so that the shape has rotational symmetry of order 2.

![Diagram](image2.png)

2. Which of these triangles are similar?

![Triangles](image3.png)

3. Which of these rectangles are similar?

![Rectangles](image4.png)

4. **ABC** is an equilateral triangle.
   - **D** lies on **BC**. **AD** is perpendicular to **BC**.
     - **a** Prove that triangle **ADC** is congruent to triangle **ADB**.
     - **b** Hence, prove that **BD = \frac{1}{2} BC**.
In the diagram, \( AB = BC = CD = DA \).
Prove that triangle \( ADB \) is congruent to triangle \( CDB \).

\[ \text{Diagram NOT accurately drawn} \]

AB is parallel to DE.
ACE and BCD are straight lines.
\( AB = 6 \text{ cm} \)
\( AC = 8 \text{ cm} \)
\( CD = 13.5 \text{ cm} \)
\( DE = 9 \text{ cm} \)

a Work out the length of CE.
b Work out the length of BC.

Parallelogram \( P \) is similar to parallelogram \( Q \).
\( \text{Calculate the value of } x \).

In triangle \( FGJ \), a line \( IH \) is drawn parallel to \( FG \).

a Prove that triangle \( HIJ \) is similar to triangle \( GFJ \).
b Calculate the length of \( HI \).
9 BE is parallel to CD.
\[ AB = 9 \text{ cm}, \ BC = 3 \text{ cm}, \ CD = 7 \text{ cm}, \ AE = 6 \text{ cm}. \]
\[ \text{a} \quad \text{Calculate the length of } ED. \\
\text{b} \quad \text{Calculate the length of } BE. \]